Further Mathematics

SAMPLE

Marking Scheme

This marking scheme has been prepared as a **guide only** to markers. This is not a set of model answers, or the exclusive answers to the questions, and there will frequently be alternative responses which will provide a valid answer. Markers are advised that, unless a question specifies that an answer be provided in a particular form, then an answer that is correct (factually or in practical terms) **must** be given the available marks.

If there is doubt as to the correctness of an answer, the relevant NCC Education materials should be the first authority.

Throughout the marking, please credit any valid alternative point.

Where markers award half marks in any part of a question, they should ensure that the total mark recorded for the question is rounded up to a whole mark.

Answer ALL questions

Marks

Question 1

a) Use the factor theorem to show that $(x-4)$ is a factor of $f(x)=2x^3-x^2-25x-12$ **2**

$\boldsymbol{f(4)=2(4)^3-(4)^2-25(4)-12=0}$ ***(M1 – evaluation of f(4))***

$\therefore (\boldsymbol{x-4})$ ***is a factor of*** $\boldsymbol{f(x)}$ ***(A1 – f(4) = 0 and conclusion stated)***

b) Hence, or otherwise, fully factorise $f(x)$. **3**

Method 1: Algebraic Long Division

$$\begin{array}{r|l} & \boldsymbol{2x^2+7x+3} \\ \hline \boldsymbol{x-4} & \boldsymbol{2x^3-x^2-25x-12} \\ & \boldsymbol{2x^3-8x^2} \\ & \qquad \boldsymbol{7x^2-25x} \\ & \qquad \boldsymbol{7x^2-28x} \\ & \qquad\qquad \boldsymbol{3x-12} \\ & \qquad\qquad \underline{\boldsymbol{3x-12}} \\ & \qquad\qquad\qquad \boldsymbol{0} \end{array}$$

(M1 – Use of Algebraic Long Divison to obtain $\boldsymbol{Ax^2+Bx+C}$ ***where*** $\boldsymbol{A,B,C\neq 0}$ ***)***

$\boldsymbol{(x-4)(2x^2+7x+3)}$ ***(A1-correct quadratic)***

$\boldsymbol{(x-4)(x+3)(2x+1)}$ ***(A1 – correct and fully factorised)***

Method 2: Equating Coefficients

$\boldsymbol{2x^3-x^2-25x-12\equiv(x-4)(Ax^2+Bx+C)}$ ***(M1-linear × quadratic with three non-zero coefficients)***

$\boldsymbol{A=2, B=7, C=3}$ ***(A1 – All three correct values)***

$\boldsymbol{(x-4)(2x^2+7x+3)\equiv(x-4)(x+3\_(2x+1)}$ ***(A1 – Fully factorised)***

Marks

c) Show that $\sum_{r=1}^{n} 4r^2 + 6r + 1 = \frac{n}{3}[4n^2 + 15n + 12]$ **4**

$\boldsymbol{\sum_{r=1}^{n} 4r^2 + 6r + 1 = 4\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1}$ ***(M1 – Separate summations)***

$\boldsymbol{= \frac{4n}{6}(n+1)(2n+1) + \frac{6n}{2}(n+1) + n}$ ***(M1 – use of summation formulae)***

$\boldsymbol{= \frac{1}{3}n[2(n+1)(2n+1) + 9(n+1) + 3]}$ ***(M1 – Factorises by n)***

$\boldsymbol{= \tfrac{1}{3}n[(4n^2 + 6n + 2) + 9n + 12]}$ ***(at least one intermediate step shown)***

$\boldsymbol{= \frac{n}{3}[4n^2 + 15n + 14]}$ ***(A1 – All correct with no errors seen)***

d) Given that 3 and $4 + i$ are roots of the cubic equation
$f(x) = x^3 - 11x^2 + ax + b = 0$

i) Find the value of a and the value of b **4**

Third root is $\boldsymbol{(4 - 3i)}$ ***(B1 – Sight of final root)***

$\boldsymbol{\left(x - (4 + 3i)\right)\left(x - (4 - 3i)\right)}$

$\boldsymbol{= x^2 - x(4 - 3i) - x(4 + 3i) + (4 + 3i)(4 - 3i)}$ ***(M1 – correct attempt at using complex roots to obtain a quadratic or cubic not containing i)***

$\boldsymbol{= x^2 - 8x + 25}$ ***(A1 – Correct quadratic obtained)***

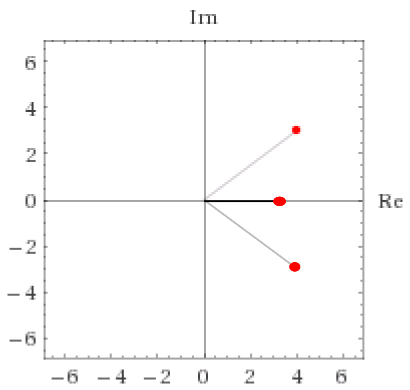
$\boldsymbol{(x - 3)(x^2 - 8x + 25) = x^3 - 8x^2 + 25x - 3x^2 + 24x - 75}$

$\boldsymbol{= x^3 - 11x^2 + 49x - 75}$

$\boldsymbol{a = 49,\ \ b = -75}$ ***(A1 – for both correct)***

Marks

ii) Show the three roots on an Argand diagram **2**



B1 – Two complex roots correct and reflection of each other in x-axis
B1 – Third root at (3, 0)

e) Given $f(x) = \frac{3(9x^2+1)}{(3x+1)(3x-1)} \equiv A + \frac{B}{3x+1} + \frac{C}{3x-1},$

i) Find the values of A, B and C **3**

$\frac{3(9x^2+1)}{(3x+1)(3x-1)} \equiv 3 + \frac{6}{(3x+1)(3x-1)}$ ***(B1)***

$\frac{6}{(3x+1)(3x-1)} \equiv \frac{A}{3x+1} + \frac{B}{3x-1} \quad \rightarrow A = -3, B = 3$ ***(M1 – Either A or B Correct)***

$\frac{3(9x^2+1)}{(3x+1)(3x-1)} \equiv 3 + \frac{3}{3x-1} - \frac{3}{3x+1}$ ***(A1 – both A and B found and all correct)***

ii) Use the quotient rule to differentiate $f(x)$, giving your answer in the form $f'(x) = \frac{-Qx}{(9x^2-1)^2}$ **2**

$$f'(x) = \frac{(3x+1)(3x-1)(54x) - 3(9x^2+1)(18x)}{(9x^2-1)^2}$$

(M1-correct un-simplified use of quotient rule)

$f'(x) = \frac{-108x}{(9x^2-1)^2}$ ***(A1 – for correct simplified quotient with Q=108)***

Total 20 Marks

Marks

Question 2

a) A curve has equation $f(x) = \frac{5}{(1+x)(3-2x)}$

i) Express $f(x)$ in terms of its partial fractions. **4**

$\frac{5}{(1+x)(3-2x)} \equiv \frac{A}{1+x} + \frac{B}{3-2x}$

$5 \equiv A(3-2x) + B(1+x)$ ***(M1)***

Let $x = \frac{3}{2} \rightarrow 5 = \frac{5B}{2} \rightarrow B = 2$ ***(A1)***

Let $x = -1 \rightarrow 5 = 5A \rightarrow A = 1$ ***(A1)***

$\therefore \frac{5}{(1+x)(3-2x)} \equiv \frac{1}{(1+x)} + \frac{2}{(3-2x)}$ ***(A1)***

b)

i) Use the binomial theorem to expand $f(x)$ in ascending powers of x, up to and including the term in x^3, simplifying each term. **5**

$\frac{1}{1+x} = (1+x)^{-1}$ ***or*** $\frac{2}{3-2x} = 2(3-2x)^{-1}$ ***(B1 for either term)***

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ ***(B1-Correct Expansion)***

$2(3-2x)^{-1} = 2 \times (3^{-1})\left(1 - \frac{2}{3}x\right)^{-1} = \frac{2}{3}\left(1 - \frac{2}{3}x\right)^{-1}$ ***(M1- Factorising)***

$2(3-2x)^{-1} = \frac{2}{3} + \frac{4}{9}x + \frac{8}{27}x^2 + \frac{16}{81}x^3 + \cdots$ ***(A1 – Correct Expansion)***

Hence, $\frac{5}{(1+x)(3-2x)} \equiv \frac{5}{3} - \frac{5}{9}x + \frac{35}{27}x^2 - \frac{65}{81}x^3$ ***(A1 – All Correct)***

ii) Determine the range of values of x for which the expansion is valid. **1**

$|x| < 1$ ***or*** $|x| < \frac{3}{2}$

Hence, valid for $|x| < 1$ ***(A1 – considers both and makes conclusion)***

Marks

c) Use your answer to a) to find $f'(x)$ and hence find the x – coordinate of the stationary point of $f(x)$ **5**

$$\frac{d}{dx}\left[\frac{5}{(1+x)(3-2x)}\right] = \frac{d}{dx}\left[(1+x)^{-1} + 2(3-2x)^{-1}\right]$$

$-(1+x)^{-2}$ $\quad or \quad$ $-2(-2)(3-2x)^{-2}$ ***(M1)***

$\frac{4}{(3-2x)^2} - \frac{1}{(1+x)^2}$ ***(A1 – Both Correct)***

$4(1+x)^2 - (3-2x)^2 = 0$

(M1 – sets derivative equal to zero and attempts to solve for x)

$4(1+2x+x^2) - (9-12x+4x^2) = 0 \rightarrow 20x - 5 = 0$ ***(M1 – Expands brackets correctly and simplifies to obtain linear equation in x)***

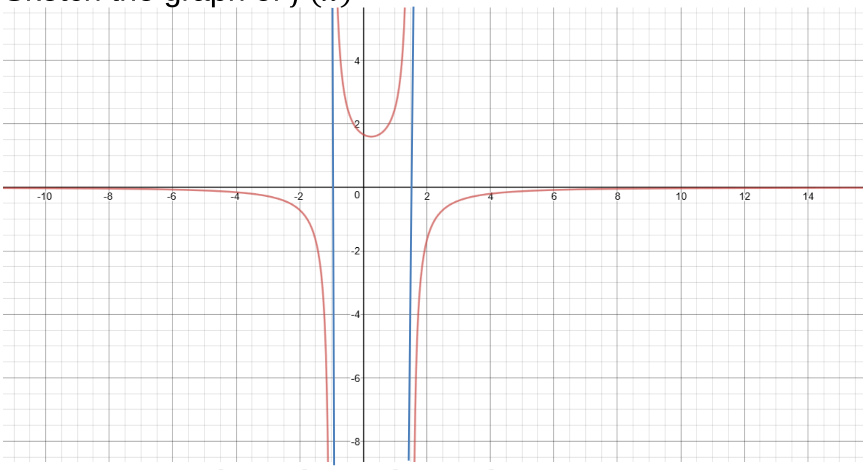
$x = \frac{1}{4}$ ***(A1)***

d) Write down the equations of all the asymptotes of f(x) **2**

Vertical Asymptotes at $x = -1$ and $x = \frac{3}{2}$ ***(B1 – both required)***

$y = 0$ ***(B1 – accept x-axis)***

e) Sketch the graph of $f(x)$ **3**



B1 - Correct shape in each quadrant
B1 - Asymptotes at x=-1 and x=3/2 correctly drawn and labelled
B1 – Minimum point at x = ¼ and 5/3 labelled on the y-axis

Total 20 Marks

Marks

Question 3

a) Given $Z_1 = 4 + \sqrt{2}i$ and $\frac{Z_1}{Z_2} = 3 + i$, find:

i) Z_2 in the form $a + ib$ **5**

$\boldsymbol{Z_2 = (4 + \sqrt{2}i) \div (3 + i)}$ ***(M1)***

(M1 multiply numerator and denominator by $\boldsymbol{(3 - i)}$
$\boldsymbol{(12 + \sqrt{2} + 3\sqrt{2}i - 4i)}$ ***(A1)***
$\boldsymbol{= \left(12 + \sqrt{2}\right) \div 10 + (3\sqrt{2} - 4)i \div 10}$ ***(A1 – Real Part , A1 – Imaginary Part)***

ii) The argument of Z_2 **2**
Arg(Z_2) ***= arctan***$\left(\frac{3\sqrt{2}-4}{12+\sqrt{2}}\right)$ ***(M1)***

Arg(Z_2) $\boldsymbol{= 0.018(086 \ldots)}$ ***(A1 to 3 s.f. or more)***

iii) The exact value of $|Z_2|$ **3**
$\boldsymbol{(12 + \sqrt{2})^2 = (12 + \sqrt{2})(12 + \sqrt{2})}$
$\boldsymbol{= 144 + 24\sqrt{2} + 2 = 146 + 24\sqrt{2}}$

$\boldsymbol{(3\sqrt{2} - 4)^2 = \left(3\sqrt{2} - 4\right)\left(3\sqrt{2} - 4\right)}$
$\boldsymbol{= 18 - 24\sqrt{2} + 16 = 34 - 24\sqrt{2}}$ ***(B1 – Either Correct)***

$\boldsymbol{|Z_2| = \sqrt{\left(146 - 24\sqrt{2}\right) + (34 + 24\sqrt{2})} \div 10}$ ***(M1 – Use of Pythagoras)***

$\boldsymbol{|Z_2| = \sqrt{1.8} = 3\sqrt{5}/5}$ ***(A1 – accept equivalent form)***

b) Given $Z_3 = -4 + 3i$ and $Z_4 = 3 + 2i$, find

i) The modulus of $(Z_4)(Z_3)^2$ **4**

$\boldsymbol{(-4)^2 + (3)^2[= 25]}$ ***(M1)***

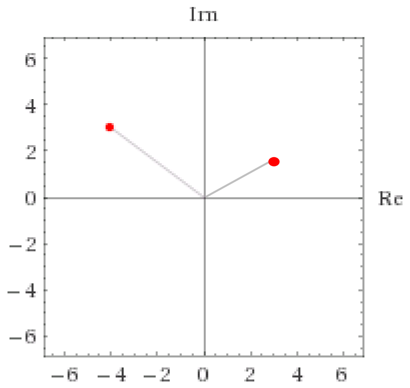
$\boldsymbol{\sqrt{((3)^2 + (2)^2)} = \sqrt{13}}$ ***(B1)***

Multiplying together (M1)

$\boldsymbol{= 25\sqrt{13} \text{ or } 90.14}$ ***(A1)***

Marks

ii) Show on an Argand diagram the points A and B, where A represents Z_3 and B represents Z_4 **2**



B1 – For one correct
B1 – For both correct

iii) Find Z_3Z_4 in polar form **2**

$$\boldsymbol{Arg\ (Z_3) = \pi - arctan\left(\frac{3}{4}\right) = 2.50^c}$$

$$\boldsymbol{Arg(Z_4) = arctan\left(\frac{2}{3}\right) = 0.59^c}$$ ***(B1 for either)***

Hence, $\boldsymbol{Z_3Z_4 = [5\sqrt{13}, 3.09]}$ ***(B1)***

iv) Determine angle $A\hat{O}B$ **1**

2.50 – 0.59 = 1.91 (*or* 109 *degrees*) ***(B1)***

v) Write Z_4 in exponential form **1**

$\boldsymbol{Z_4 = \sqrt{13}e^{0.59i}}$ ***(B1)***

Total 20 Marks

Marks

Question 4

a) Prove that $cosh^2(x) - sinh^2(x) \equiv 1$ using the exponential definition for *cosh*(x) and *sinh*(x) **2**

$\left(\frac{e^x+e^{-x}}{2}\right)^2 = \frac{e^{2x}+e^{-2x}+2}{4}$ $\left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+e^{-2x}-2}{4}$ ***(M1 – Use correct exponential forms of cosh(x) and sinh(x) and both brackets squared)***

A1 – Finishes proof with no errors seen.

b) Show that the equation $tan^2(x) = 3\sec(x) + 9 = 0$ can be written as $sec^2(x) - 3\sec(x) - 10 = 0$ **2**

$sec^2(x) - 1 = 3sec(x) + 9$ ***(M1 – Uses correct identity $1 + tan^2(x) \equiv sec^2(x)$)***

$sec^2(x) - 3sec(x) - 10 = 0$ ***(A1 – without wrong working)***

c) Hence, solve $tan^2(x) = 3\sec(x) + 9$ giving all values of x in the interval $0 \leq x \leq 360$ giving solutions to 2 d.p. where necessary **4**

$(sec(x) + 2)(sec(x) - 5) = 0$ ***(M1 – correct factorisation)***

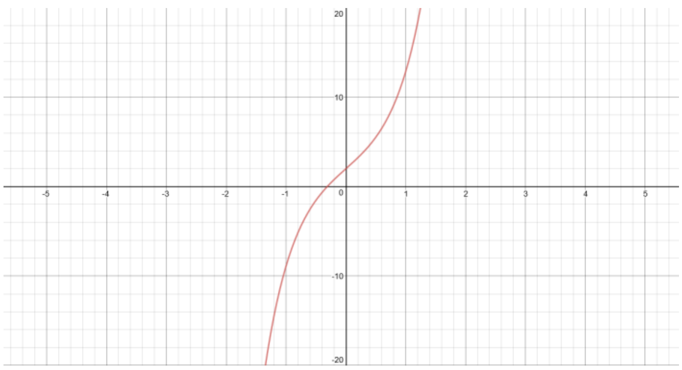
$sec(x) = 5 \ \ or \ \ sec(x) = -2$ ***(A1 – Both)***

$\frac{1}{cos(x)} = 5 \rightarrow cos(x) = \frac{1}{5}$

$\frac{1}{cos(x)} = -2 \rightarrow cos(x) = -\frac{1}{2}$

$x = 78.46^o, 281.54^o, 120^o, 240^o$ ***(A1 – two correct, A2 – all four correct)***

d) Sketch the graph of $y = 3\sinh(2x) + 2$ **2**



B1 – Correct shape and in correct quadrants
B1 – Passes through (0,2)

Marks

e) Prove that $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ **4**

$\boldsymbol{e^{ix} = cos(x) + isin(x)}$ ***(B1 – correct use of Euler's Formula)***

$\boldsymbol{e^{i(-x)} = cos(-x) + isin(-x)}$
$\boldsymbol{= cos(x) - isin(x)}$ ***(M1 – Use of odd and even functions)***

$\boldsymbol{e^{ix} - e^{-ix} = 2isin(x)}$ ***(M1 – Subtracting either way round)***

$\boldsymbol{sin(x) = \frac{e^{ix} - e^{-ix}}{2i}}$ ***(A1 – All correct with no errors seen)***

f) Find the first three terms in the series expansion of cos(*x*) using Maclaurin's expansion **2**

$\boldsymbol{f(x) = cos(x) \rightarrow f(0) = 1}$

$\boldsymbol{f'(x) = -sin(x) \rightarrow f'(0) = 0}$

$\boldsymbol{f''(x) = -cos(x) \rightarrow f''(0) = -1}$

$\boldsymbol{f'''(x) = sin(x) \rightarrow f'''(0) = 0}$

$\boldsymbol{f'^{v}(x) = cos(x) \rightarrow f'^{v}(0) = 1}$

(M1 – correct values from differentiation – allow one error)

Hence, $\boldsymbol{cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}}$ ***(A1 – All correct)***

i) Given $(1 + x)^{-1} \approx 1 - x + x^2$, find the first three non-zero terms in the expansion of sec(x), in ascending order. **4**

$\boldsymbol{sec(x) = \frac{1}{cos(x)} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24}}}$ ***(B1 – FT from previous question)***

$\boldsymbol{= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^{-1}}$

$\boldsymbol{= \left\{1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2\right\}}$ ***(M1 – Uses given result)***

$\boldsymbol{= \left\{1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \cdots\right\}}$

$\boldsymbol{= 1 + \frac{x^2}{2} + \frac{5x^4}{24}}$ ***(A1 – first two terms correct)***
(A1 – last term correct)

Total 20 Marks

Marks

Question 5

a) Describe the transformation represented by the matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ **2**

B1 – Rotation
B1 - 45^o clockwise about the origin

i) Find A^2 **2**

$$A^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ ***(M1 – Decent Attempt At Multiplying)***

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ ***(A1 - All Correct)***

ii) Describe the transformation represented by A^2 **1**

Rotation 90^o clockwise about the origin (A1 – all required)

b) Given $A = \begin{pmatrix} a & -6 \\ 2 & a+6 \end{pmatrix}$

i) Find det(A) giving your answer in terms of a. **2**

$$det(A) = a(a+6) - (-6 \times 2)$$ ***(M1 – correct use of determinant)***

$$det(A) = a^2 + 6a + 12$$ ***(A1)***

ii) Show that the matrix A is non-singular for all values of a. **2**

$$a^2 + 6a + 12 \equiv (a+3)^2 + 3$$ ***(M1 – Attempts to complete the square)***

$(a+3)^2 + 3 > 0 \quad \forall x$ ***, hence $det(A) \neq 0$ (A1 – Needs Conclusion)***

iii) Find A^{-1}, when a=3 **2**

$$A = \begin{pmatrix} 3 & -6 \\ 2 & 9 \end{pmatrix} \rightarrow det(A) = 27 + 12 = 39$$ ***(M1)***

$$A^{-1} = \frac{1}{39}\begin{pmatrix} 9 & 6 \\ -2 & 3 \end{pmatrix}$$ ***(A1 – All Correct)***

Marks

c) A rectangular hyperbola H has parametric equations given by $x = 3t, y = \frac{3}{t}, t \neq 0$. **3**

i) The line L has equation $6y = 4x - 15$. Show that L intersects H when $4t^2 - 5t - 6 = 0$

$\boldsymbol{H: x = 3t, y = \frac{3}{t}}$

$\boldsymbol{L: 6y = 4x - 15 \;\rightarrow\; 6\left(\frac{3}{t}\right) = 4(3t) - 15 \;\rightarrow\; 18 = 12t^2 - 15t}$ ***(M1- Attempts to substitute x and y into line – A1 – all correct)***

$\boldsymbol{12t^2 - 15t - 18 = 0 \;\rightarrow\; 4t^2 - 5t - 6 = 0}$ ***(A1)***

ii) Find $\frac{dy}{dx}$ at the point where t=2 **3**

$\boldsymbol{\frac{dx}{dt} = 3, \frac{dy}{dt} = -\frac{3}{t^2}}$ ***(M1 – for both)***

$\boldsymbol{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{3}{t^2} \times \frac{1}{3}}$ ***(M1 – Use of the chain rule)***

At $\boldsymbol{t = 2, \frac{dy}{dx} = -\frac{3}{4} \times \frac{1}{3} = -\frac{1}{4}}$ ***(A1)***

ii) The line intersects the hyperbola at two points, a and b. Find the coordinates of a and the coordinates of b. **3**

$\boldsymbol{(t-2)(4t+3) = 0 \;\rightarrow\; t = 2, t = -\frac{3}{4}}$ ***(M1 – factorises quadratic in t)***

When $\boldsymbol{t = 2, x = 6, y = \frac{3}{2} \;\rightarrow\; \left(6, \frac{3}{2}\right)}$ ***(A1)***

When $\boldsymbol{t = -\frac{3}{4}, x = -\frac{9}{4}, y = -4 \;\rightarrow\; \left(-\frac{9}{4}, -4\right)}$ ***(A1)***

Total 20 Marks

End of paper

Marks

Learning Outcomes matrix

Question	Learning Outcomes assessed	Marker can differentiate between varying levels of achievement
1	1, 2, 5, 6	Yes
2	1,4,6	Yes
3	2,8	Yes
4	6,7,8	Yes
5	3,4	Yes

Grade descriptors

Learning Outcome	Pass	Merit	Distinction
Understand different techniques to solve cubic equations and write expressions in terms of their partial fractions	Demonstrate adequate understanding of techniques	Demonstrate robust understanding of techniques	Demonstrate highly comprehensive understanding of techniques
Be able to work with complex numbers, perform arithmetic calculations using complex numbers, solve higher order polynomials with complex roots and sketch regions in the complex plane	Demonstrate ability to perform the tasks	Demonstrate ability to perform the tasks consistently well	Demonstrate ability to perform the tasks to the highest standard
Be able to perform arithmetic operations using matrices, understand basic transformations using matrices and, in addition, understand which matrices represent linear transformations and calculate the inverse of a matrix	Demonstrate ability to perform techniques	Demonstrate ability to perform techniques consistently well	Demonstrate ability to perform techniques to the highest standard
Understand the properties of rational functions and understand conic sections	Demonstrate adequate understanding of techniques	Demonstrate robust understanding of techniques	Demonstrate highly comprehensive understanding of techniques
Understand how to use sigma notation to calculate the sum of simple finite series, and appreciate the relationship between the roots of polynomials and their coefficients	Demonstrate adequate understanding of techniques	Demonstrate robust understanding of techniques	Demonstrate highly comprehensive understanding of techniques

Grade descriptors continue on next page

Marks

Learning Outcome	Pass	Merit	Distinction
Understand further techniques in calculus to differentiate combinations of functions, how to use these techniques to solve problems involving functions given parametrically and how to derive Maclaurin and Taylor series	Demonstrate adequate understanding of techniques	Demonstrate robust understanding of techniques	Demonstrate highly comprehensive understanding of techniques
Understand further trigonometry and hyperbolic functions	Demonstrate adequate understanding of techniques	Demonstrate robust understanding of techniques	Demonstrate highly comprehensive understanding of techniques
Understand Euler's relation and De Moivre's theorem and derive relations between trigonometric functions and hyperbolic functions	Demonstrate adequate level of understanding	Demonstrate robust level of understanding	Demonstrate highly comprehensive level of understanding